ON THE THEORY OF DIFFERENTIAL GAMES OF SYSTEMS WITH AFTEREFFECT

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The game problem on the minimax (maximin) of the time to encounter with a given closed set is considered for systems with aftereffect. The problem is investigated on the basis of extremal strategies whose construction is based on the notion of absorption of the target by the controlled process. This notion was introduced in [1] for systems described by ordinary differential equations. The results are applied to linear systems in connection with the extremal aiming rule for systems described by ordinary differential equations [2]. The present paper is directly related to studies [1−12].

1. Let a controlled system with aftereffect be described by the equation

$$dx(t) / dt = f_1(t, x_t(s), u) + f_2(t, x_t(s), v) \tag{1.1}$$

Here x is the n-dimensional phase coordinate vector; the r_1-dimensional vector u and the r_2-dimensional vecter v are the controlling forces at the disposal of the first and second player, respectively. These vectors are subject to the restrictions

$$u \in P, \qquad v \in Q \tag{1.2}$$

where P, Q are bounded closed sets; the functionals $f_i(t, x(s), w)$ are continuous and satisfy the Lipschitz conditions with respect to the functions $x(s)$, $-\tau \leqslant s \leqslant 0$. (A detailed description of system (1.1) will be found in [9, 10] along with definitions of some of the notions and symbols occurring below).

In [10] the game problem of guiding system motions onto a given closed set M was considered for system (1.1). We now propose to use the results of [10] to investigate the game on the minimax (maximin) of the time to encounter of system (1.1) with the target M.

This game is as follows. The initial game position $p_0 = \{t_0, x_0(s)\}$, $(t \in [t_\alpha, t_\beta))$, $x(s) \in C_{[-\tau, 0]}$ is given. The first player strives by suitable choice of strategy (see [9]) to bring the motions $x[t] = x[t, p_0, U, V_T]$ onto the set M in the shortest possible time. The second player strives by suitable choice of the strategy V to prevent encounter of motions $x[t] = x[t, p_0, U_T, V]$ with the target M or at least to maximize the time until encounter occurs.

Let us agree on the following notation: wishing to emphasize that we are referring to some motion $x[t]$ of system (1.1) and not to the position $x[t]$ of this system at an instant t, we denote the former by $x[\cdot]$. For example, $x[\cdot] = x[\cdot, p_0, U, V_T]$ is the motion of system (1.1) from the position p_0 which corresponds to the strategies U and V_T (see [9]). Let us refine our statement of the problem. The result of the first player's actions in the course of the game are estimated by the quantity

$$\gamma_1(U) = \sup_{x[\cdot]} \nu(x[\cdot]) \tag{1.3}$$

where $x[\cdot] \in \{x[\cdot, p_0, U, V_T]\}$, $\vartheta(x[\cdot])$ is the first instant of encounter of the motion $x[\cdot]$ with the target M. (We set $\vartheta = \infty$ in (1.3) if such an encounter does not occur).

Definitions. (1°). A strategy U° is called the minimax strategy if

$$\gamma_1(U^\circ) = \min_U \gamma_1(U) \tag{1.4}$$

where min is taken over all the first-player strategies U

(2°). A number $\gamma_0 = \gamma_1(U^\circ)$ is called the value of the game if

$$\gamma_1(U^\circ) = \sup_{\varepsilon>0} \sup_V \inf_{x[\cdot]} \vartheta^\varepsilon(x[\cdot]) \tag{1.5}$$

where $x[\cdot] \in \{x[\cdot, p_0, U_T, V]\}$, $\vartheta^\varepsilon(x[\cdot])$ is the initial instant of arrival of the motion $x[\cdot]$ in a closed ε-neighborhood M_ε of the set M; $\sup_V$ is taken over all the second-player strategies V.

(3°). A sequence $\{V_j^\circ\}$ of strategies for which the condition

$$\lim_{j\to\infty} \inf_{x[\cdot]} \vartheta^{\varepsilon_j}(x[\cdot]) = \gamma_0 \tag{1.6}$$

is fulfilled (where $x[\cdot] \in \{x[\cdot, p_0, U_T, V_j^\circ]\}$, $\varepsilon_j = \text{const} > 0$ $(j = 1, 2, \ldots)$ will be called the maximum sequence of second-player strategies.

In the present paper we prove the existence of a minimax first-player strategy U° and of the value of the game; we also investigate the structure of the strategy U° and of a maximin sequence of strategies V_j°.

2. In this section we investigate the properties of the sets of positional absorption of the target M by system (1.1) by a given instant (see [9]) which are necessary for the discussion to follow. For convenience we recall the following definitions from [9]. Let some game position $p_* = \{t_*, x_*(s)\}$ be given. We denote the distance from x to M in E_n by $\rho(x, M)$.

Definitions. (4°). System (1.1) absorbs the target M from the position p_* by the instant ϑ positionally if

$$\sup_V \inf_{x[\cdot]} \min_{t_* \leqslant t \leqslant \vartheta} \rho(x[t], M) = 0 \tag{2.1}$$

where $x[\cdot] \in \{x[\cdot, p_0, U_T, V]\}$, sup is taken over all the second-player strategies V.

(5°). The collection of all functions $x(s) \in C_{[-\tau, 0]}$ such that from the position $p = \{t, x(s)\}$ system (1.1) absorbs M positionally by the instant ϑ is called the set of positional absorption of the target M by the instant ϑ and is denoted by the symbol $W(t, \vartheta)$.

Lemma 2.1. The set $W(t_*, \vartheta)$ is closed in the space $C_{[-\tau, 0]}$.

Proof. We shall prove the lemma by reductio ad absurdum. We begin by assuming that there exists a sequence $\{x^{(k)}(s)\}$ of elements $x^{(k)}(s) \in W(t_*, \vartheta)$ which converges in $C_{[-\tau,0]}$ to the element $x_*(s) \notin W(t_*, \vartheta)$. We denote by $M(t', t'')$, $X(p, t'', U_T, V)$ sets from the space $E_{n+1}\{y\}$ which are of the form

$$M(t', t'') = \{y = (t_1, z) \mid t_1 \in [t', t''], z \in M\}$$

$$X(p, t'', U_T, V) = \{y = (t_1, x[t_1]) \mid t_1 \in [t', t'']$$

$$x[t_1] \in \{x[t_1, p, U_T, V]\}, p = \{t', x_*(s)\}\}$$

The closure of the set M in E_n and of the set $\{x[\cdot] = x[\cdot, p, U_T, V]\}$ in $C_{[t',t'']}$ (the latter follows directly from the definition of the motions $x[\cdot, p, U_T, V]$)

implies the closure of the sets $M(t', t'')$, $X(p, t'', U_T, V)$ in E_{n+1}. Since $x_*(s) \notin W(t_*, \vartheta)$, by Definition (5*) there exists a second-player strategy V_0 such that the closed set $X(p_*, \vartheta, U_T, V_0)$, $(p_* = \{t_*, x_*(s)\})$ does not intersect the closed set $M(t_*, \vartheta)$. Hence, some closed α-neighborhood ($\alpha > 0$) $X_\alpha(p_*, \vartheta, U_T, V_0)$ of the set $X(p_*, \vartheta, U_T, V_0)$ also does not intersect $M(t_*, \vartheta)$.

Now let us consider the system of sets $Y_t \subset C_{[-\tau, 0]}$ of the form

$$Y_t = \{x(s) = x[t+s] \mid x[t+s] \in \{x[t+s, p_*, U_T, V_0]\}\}$$

in the interval $[t_*, \vartheta]$. Let us show that this system of sets is strongly v-stable (see [9]). Let t_1 be an arbitrary instant from $[t_*, \vartheta]$, let $x_1(s)$ be an arbitrary element of Y_t, and let U_u be an arbitrary first-player strategy. Clearly, we need only show that in the space $C_{[t_1, \vartheta]}$ the intersection of the sets $Q_1 = \{x[t, p_1, U_u, V_T]\}$ and $Q_2 = \{x[t, p_1, U_T, V_0]\}$, where $p_1 = \{t_1, x_1(s)\}$, is nonempty. Let us consider in $C_{[t_1, \vartheta]}$ a sequence $\{x[t]_{\Delta j}\}$ $\{\delta_j\} \to 0$ of functions such that $x(t_1 + s]_{\Delta_j} = x_1(s)$ and

$$dx[t]_{\Delta_j}/dt = f_1(t, x_t[s]_{\Delta_j}, u(t)) + f_2(t, x_t[s]_{\Delta_j}, v_j[t])$$

for almost all $t \in [t_1, \vartheta]$. Here $u(t)$ is the program control which generates the strategy U_u; $v_j[t] = v_j[\tau_i] \in V_0(\tau_i, x_{\tau i}[s]_{\Delta_j})$, $\tau_i \leqslant t < \tau_{i+1}$ $(\tau_i = \tau_i(j))$; $V_0(t, x(s))$ is the set which defines the strategy V_0. We denote by $F_i(t, x(s))$ $(i = 1, 2)$ the convex shell of the set of vectors $\{f_i(t, x(s), w_i) \mid w_i \in W_i\}$ $(W_1 = P, W_2 = Q)$. By virtue of the bicompactness of the set of solutions of the equation in contingencies

$$dx(s)/dt \in f_1(t, x_t(s), u(t)) + F_2(t, x_t(s)) \tag{2.2}$$

we can (choosing a subsequence if necessary) assume that the sequence $\{x[t]_{\Delta_j}\}$ converges to some solution $x^\circ[t]$ of Eq. (2.3). (The bicompactness of the collection of solutions of Eq. (2.2) can be proved just as the bicompactness in $C_{[t_*, t^*]}$ of the set of solutions of Eq. (2.3) (see below)). The construction of the function $x^\circ[t]$ and the definitions of the motions $x[t, p, U_T, V]$ yield the following relation: $x^\circ[t] \in Q_1 \cap Q_2$.

Thus, the system of sets Y_t, $t_* \leqslant t \leqslant \vartheta$ is strongly v-stable. Choosing a sufficiently large k and proceeding on the basis of Lemma 2.4 of [10], we find that the second-player strategy V^e extremal to this system of sets ensures the inclusion

$$X(p_k, \vartheta, U_T, V^e) \subset X_\alpha(p_*, \vartheta, U_T, V_0)$$

where $p_k = \{t_* \; x^{(k)}(s)\}$. By virtue of the condition of nonintersection of the sets $X_\alpha(p_*, \vartheta, U_T, V_0)$, $M(t_*, \vartheta)$, the latter contradicts Definition (5*) of the set $W(t_*, \vartheta)$. The lemma has been proved.

Lemma 2.2. The system of sets $W(t, \vartheta)$, $t_0 \leqslant t \leqslant \vartheta$, is u-stable.

Proof. Let us assume the contrary, (see [9]), i.e. that there exist instants t_*, t^* from $[t_0, \vartheta]$, $(t^* > t_*)$, an element $x_*(s) \in W(t_*, \vartheta)$, and a strategy V_v such that the following Conditions are fulfilled simultaneously: (1) the sets Q^* and $W(t^*, \vartheta)$ do not intersect; (2) the sets $X(p_*, \vartheta, U_T, V_v)$ and $M(t_*, t^*)$ do not intersect. Here $Q^* = \{q(s) = x[t^* + s] \mid x[t^* + s] \in \{x[t^* + s, p_*, U_T, V_v]\}\} \subset C_{[-\tau, 0]}$; $X(p_*, \vartheta, U_T, V_v)$ are defined similarly to the sets $X(p_*, \vartheta, U_T, V)$ (see above); $p_* = \{t_*, x_*(s)\}$.

It is important to note that the set Q^* is a bicompact in the space $C_{[-\tau, 0]}$. In fact, proceeding on the basis of the continuity of the functionals $f_i(t, x(s), w)$ from (1.1), the Lipschitz condition for these functionals with respect to $x(s)$, and the properties of the sets P, Q, we infer from (1.2) in the usual way (see [13]) that the set $\{x[t]\} = \{x[t, p_*, U_T, V_v]\}$ of solutions of the equation

$$dx[t] / dt \in F_1(t, x_t[s]) + f_2(t, x_t[s], v(t))$$
$$x[t_* + s] = x_*(s) \qquad (2.3)$$

is uniformly bounded and equicontinuous on $[t_*, t^*]$. Let us show that this set is closed in $C_{[t_*, t^*]}$ (see [9]).

Let $x[t]$ be an arbitrary convergent sequence of elements from $\{x[t, p_*, U_T, V_v]\}$ and let $x^\circ[t]$ be its limit. Since the absolutely continuous functions $x^{(k)}[t]$ satisfy the Lipschitz condition equicontinuously in the interval $[t_*, t^*]$, it follows that the limit function $x^\circ[t]$, satisfies this condition and is therefore absolutely continuous on $[t_*, t^*]$.

By virtue of the closure in E_n of the set $\psi(t, x(s)) = F_1(t, x(s)) + f_2(t, x(s), v(t))$ (which follows from the continuity of f_1 and the closure of P), we need merely verify that for almost all $t \in [t_*, t^*]$ the vector $dx^\circ[t] / dt$ belongs to an arbitrary closed α-neighborhood $\psi_\alpha(t, x_t^\circ[s])$ of the set $\psi(t, x_t^\circ[s])$.

Let $\varkappa \in [t_*, t^*]$ be an arbitrary point, where $dx^\circ[t] / dt$ exists. By virtue of the summability of $v(t)$ in $[t_*, t^*]$ we can assume that the point $\varkappa$ is the Lebesgue point of the function $v(t)$. The definition of a derivative, the uniform convergence of $\{x^{(k)}[t]\}$ to $x^\circ[t]$, the continuity of the functionals $f_i(t, x(s), w)$, and the Lipschitz condition for them with respect to $x(s)$ imply that for any number $\alpha > 0$ there exist positive numbers $\Delta_0 (\Delta_0 \leqslant t - \varkappa)$ and k_1 such that for $k > k_1$, $\Delta \leqslant \Delta_0$ we have

$$\frac{dx^\circ[\varkappa]}{dt} = \frac{1}{\Delta}\int_{\varkappa}^{\varkappa+\Delta} [\varphi_k(\xi) + f_2(\xi, x_\xi^0[s], v(\xi))]\, d\xi + w_1(\Delta) + w_2(\Delta, k_1) \qquad (2.4)$$

where $\varphi_k(\xi) \in F_1(\xi, x_\xi^{(k)}[s])$ is a summable function, and $\|w_1(\Delta)\| \leqslant \alpha / 3$, $w_2(\Delta, k_1)\| \leqslant \alpha / 3$.

Recalling that the set $F_1(\xi, x(s))$ is semicontinuous above relative to inclusion with respect to $x(s)$, that the sequence $\{x^{(k)}[t]\}$ converges uniformly to $x^\circ[t]$, and that $\varkappa$ is a Lebesgue point of the function $v(t)$, we conclude that the number $\alpha / 3$ is associated with numbers $\Delta_1 > 0$, $k_2 \geqslant k_1$ such that the vector defined by the first term in the right side of (2.4) is contained in the set $\Psi_{\alpha/3}(\varkappa, \overset{\circ}{x}_\varkappa[s])$, provided that $\Delta \leqslant \Delta_1$, $k \geqslant k_2$. By virtue of the convexity and closure of the set $\psi(t, x(s))$ and the arbitrariness of $\varkappa$, the above statements yield the required inclusion: the vector $dx^\circ[t] / dt \in \Psi(t, x_t^\circ[s])$ for almost all $t \in [t_*, t^*]$.

The proven bicompactness in $C_{[t_*, t^*]}$ of the set of solutions $\{x[t, p^*, U_T, V_v]\}$ and the definition of the element $q(s) = x[t_* + s, p_*, U_T, V_v]$ imply that the set Q^* is bicompact in $C_{[-\tau, 0]}$.

Let us take an arbitrary element $q_k(s) \in Q^*$. By the definition of the set $W(t, \vartheta)$ under Condition (1) there exists a second-player strategy $V = V(q_k)$ such that the set $M(t^*, \vartheta)$ does not intersect the set $X(p_k, \vartheta, U_T, V(q_k))$, where $p_k = \{t^*, q_k(s)\}$.

Let us consider the system of sets

$$\Gamma_t = \{x(s) = x[t+s] \mid x[t+s] \in \{x[t+s, p_k, U_T, V(q_k)]\}\}$$

in the interval $[t^*, \vartheta]$.

Arguments similar to those above can be used to show that this system of sets is strongly v-stable. But then, proceeding from Lemma 2.4 of [10] and recalling the closure of the sets $X(p_k, \vartheta, U_T, V(q_k))$, $W(t, \vartheta)$, $M(t^*, \vartheta)$, we can say that there exist positive numbers α_k, β_k such that the set

$$Z_k = \bigcup_{q \in S(\beta_k)} X(p, \vartheta, U_T, V^e(q_k))$$

does not intersect a closed α_k-neighborhood of the set $M(t^*, \vartheta)$. Here $V^e(q_k)$ is a second-player strategy extremal (see [9, 10]) to the system of sets Γ_t, $t^* \leqslant t \leqslant \vartheta$ and $p = \{t^*, q(s)\}$, $q(s) \in Q^*$; $S(\beta_k)$ is a neighborhood of radius β_k in $C_{[-\tau, 0]}$ of the element $q_k(s)$. By virtue of the proven bicompactness of the set Q^* this entire set can be covered by a finite system of such neighborhoods $S(\beta_k)$, $k = 1, \ldots, N$. But the foregoing then implies that the set Z, which is the union of the sets Z_k, $k = 1, \ldots, N$, does not intersect the closed α-neighborhood $M_\alpha(t^*, \vartheta)$ of the set $M(t^*, \vartheta)$, where $\alpha = \min\{\alpha_1, \ldots, \alpha_N\}$.

Now let us consider on the segment $[t_*, \vartheta]$ the system of sets from the space $C_{[-\tau, 0]}$ which is of the form

$$B_t = \{x(s) = x[t+s] \mid x[t+s] \in \{x[t+s, p_*, U_T, V_v]\}\}$$

for $t \in [t_*, t^*]$, and

$$B_t = \{x(s) = x[t+s] \mid x[t+s] \in \{x[t+s, p, U_T, V^e(q_k)]\}\}$$

for $t \in [t^*, \vartheta]$, where

$$p = \{t^*, q(s)\}, \quad q(s) \in Q^*; \quad \{x[\cdot, p, U_T, V^e(q_k)]\}$$

represents the collection of all motions of system (1.1) which generate the set Z constructed above. The definition of the system of sets B_t implies that this system is strongly v-stable (see the arguments above). Since $x_*(s) \in B_{t_*}$, it follows by Lemma 2.3 [10] that the second-player strategy V^e extremal to the sets B_t, $t_* \leqslant t \leqslant \vartheta$ satisfies the condition

$$\inf_{x(s) \in B_t} \max_{-\tau \leqslant s \leqslant 0} \|x_t[s] - x(s)\| = 0, \qquad t_* \leqslant t \leqslant \vartheta \tag{2.5}$$

where $x[t]$ is any motion $x[t, p_*, U_T, V^e]$.

Since the sets Z and $M(t^*, \vartheta)$ do not intersect, and since Condition (2) and the inclusion $x_*(s) \in W(t_*, \vartheta)$, it follows that Eq. (2.5) contradicts the definition of the positional absorption set $W(t_*, \vartheta)$. The lemma has been proved.

Lemma 2.3. The set $W(t_0, \vartheta)$ is semicontinuous above relative to inclusion with respect to ϑ.

Proof. Let $\vartheta = \vartheta_*$ be an arbitrary instant larger than t_0. We must show that for any arbitrarily small positive number $\alpha > 0$ there exists a positive number $\delta = \delta(\alpha)$ such that the inclusion $W(t_0, \vartheta) \subset W_\alpha(t_0, \vartheta_*)$ is fulfilled for all ϑ which satisfy the inequality $|\vartheta_* - \vartheta| \leqslant \delta$ $(\vartheta \geqslant t_0)$. Here $W_\alpha(t_0, \vartheta_*)$ is an α-neighborhood in $C_{[-\tau, 0]}$ of the set $W(t_0, \vartheta_*)$, i.e. of the collection of elements $y(s) \in C_{[-\tau, 0]}$ of the form $y(s) = x(s) + z(s)$, where

$$x(s) \in W(t_0, \vartheta_*), \quad \|z(s)\|_\tau = \max_s \|z(s)\| < \alpha.$$

Let us assume the contrary, i.e. that there exists an α_0-neighborhood of the set $W(t_0, \vartheta_*)$ such that however close the instant ϑ is to ϑ_*, the set $W(t_0, \vartheta)$ contains an element $x_*(s)$ which does not belong to $W_\alpha(t_0, \vartheta_*)$. By virtue of the inclusion

$W(t, \vartheta) \subset W(t, \vartheta_*)$ for $\vartheta \leqslant \vartheta_*$ which follows from the definition of a positional absorption set, the only possibility here is that $\vartheta > \vartheta_*$. Since $x_*(s) \notin W(t_0, \vartheta_*)$, the second player has at his disposal a strategy V_0 such that the closed set $D(\vartheta_*) = X(p_*, \vartheta_*, U_T, V_0)$, where $p_* = \{t_0, x_*(s)\}$, does not intersect the closed set $M(t_0, \vartheta_*)$ (see above). But then some closed ε-neighborhood $D_\varepsilon(\vartheta_*)$ of the set $D(\vartheta_*)$ does not intersect the set $M(t_0, \vartheta_* + \varepsilon)$. Further, proceeding from the definitions of the motions $x[t, p_0, U_T, V]$ (see [9]) and of the set $D(\vartheta_*)$, we can verify directly that whatever the positive number ε and the instant $\vartheta^* \geqslant t_0$, there exists a number $\beta_0 > 0$ such that the inclusion $D(\vartheta^* + \beta) \subset D_\varepsilon(\vartheta^*)$ is valid for $\beta \leqslant \beta_0$ Setting $\vartheta^* = \vartheta_*$, $\varepsilon \leqslant \vartheta$, choosing a number β sufficiently small to ensure that $\vartheta_* + \beta = \vartheta$, and recalling that the sets $D_\varepsilon(\vartheta_*)$ and $M(t_0, \vartheta_* + \varepsilon)$ do not intersect, we find that the set $D(\vartheta)$ also does not intersect $M(t_0, \vartheta_*, + \varepsilon)$. This contradicts the condition $x_*(s) \in W(t_0, \vartheta)$.

3. Now let us consider the game on the minimax (maximin) of the time until encounter of system (1.1) with the target M.

Let the initial game position $p_0 = \{t_0, x_0(s)\}$ be such that

$$x_0(s) \in W(t_0, \vartheta) \tag{3.1}$$

for some finite ϑ. Then there exists a smallest instant of positional absorption of the target M by system (1.1) from the position p_0, i.e. a smallest number ϑ satisfying condition (3.1). In fact, let us denote the exact lower bound of this set of numbers $\{\vartheta\}$ by $\vartheta_0 = \vartheta_0(p_0)$. Assuming that ϑ_0 is not contained in $\{\vartheta\}$, we conclude that $x_0(s)$ is not contained in the set $W(t_0, \vartheta_0)$. The latter cannot happen by virtue of Lemma 2.3 and condition (3.1).

Now let $\{\alpha_j\}$ be an arbitrary sequence of positive numbers $\vartheta_j = \vartheta_0 - \alpha$ which converges to zero. Then $x_0(s) \in W(t, \vartheta_j)$, so that, by the definition of the sets $W(t, \vartheta)$, there exists a second-player strategy V_j which guarantees that the motions $x[t, p_0, U_T, V_0]$ of system will not hit some closed ε_j-neighborhood of the set M for any $t \in [t_0, \vartheta_j]$. Bearing in mind what we have said and proceeding from Theorem 2.2 of [10], Lemma 2.2, and Definitions (1°) and (3°), we obtain the following statement.

Theorem 3.1. Let the initial game position $p_0 = \{t_0, x_0(s)\}$ satisfy inclusion (3.1) for at least one $\vartheta < \infty$. Then there exists a smallest instant ϑ_0 of positional absorption of the target M by system (1.1). The first-player strategy U^e extremal to the system of sets $W(t, \vartheta_0)$, $t_0 \leqslant t \leqslant \vartheta_0$, ensures the condition

$$\gamma_1(U^e) \leqslant \vartheta_0$$

The strategy U^e is minimax, ϑ_0 is the value of the game, and the sequence $\{V_j\}$ of strategies is maximin.

Note 3.1. It is sometimes convenient to construct the maximin sequence $\{V_j^\circ\}$ as follows (see [3]). Let $\Gamma(t_*, \vartheta_j)$ be a set of functions $x(s) \in C_{[-\tau, 0]}$ such that the following condition is fulfilled for motions $x[\cdot] = x[\cdot, p, U, V_T]$ of system (1.1) from the position $p = \{t_*, x(s)\}$:

$$\inf_U \sup \min_{t_* \leqslant t \leqslant \vartheta_j} \rho(x[t], M) \geqslant \varepsilon_j \tag{3.2}$$

where inf is taken over all the first-player strategies U where ε_j is sufficiently small, and $x[\cdot] \in \{x[\cdot, p, U, V_T]\}$.

Arguments similar to those used to prove Lemmas 2.1 and 2.2 can be used to verify the following statement.

Lemma 3.1. The sets $\Gamma(t, \vartheta_j)$ are closed for any $t \in [t_0, \vartheta_j]$. The system of sets $\Gamma(t, \vartheta_j)$, $t_0 \leqslant t \leqslant \vartheta_j$ is strongly v-stable.

Lemmas 2.3 [10] and 3.1 imply that the second-player strategy V_j° extremal to the system of sets $\Gamma(t, \vartheta_j)$, $t_0 \leqslant t \leqslant \vartheta_j$ ensures fulfillment of the condition

$$\inf_{x(\cdot)\in\Gamma(t,\vartheta_j)} \max_{-\tau\leqslant s\leqslant 0} \|x_t[s] - x(s)\| = 0$$

where $x[t]$ is any motion $x[t, p_0, U_T, V_j^e]$. The sequence $\{V_j^e\}$ is clearly maximin.

Note 3.1. The results presented in Sects. 2 and 3 remain valid if the set M varies continuously with time, i.e. if $M = M(t)$.

4. The application to specific nonlinear systems of the above method for constructing the minimax first-player strategy U° and the maximin sequence $\{V_j^\circ\}$ of second-player strategies entails difficulties having to do with the construction of the sets $W(t, \vartheta)$ and the determination of the instant of absorption ϑ_0.

However, in this case of linear system (1.1) we can follow the ideas of the monograph [2] (as in the case of systems described by ordinary differential equations [8]) to determine the effective conditions of absorption of the target M by the controlled motion, and also the conditions of deviation from this target. In place of the general procedure for constructing the strategies U° and V_j° described above we can make use of the extremal aiming rule [1, 2]. The basic results obtainable in this way are described briefly below.

Let system (1.1) be of the form

$$\frac{dx(t)}{dt} = \int_{-\tau}^{0} dA(t,s)\, x_t(s) + B(t)u - C(t)v \tag{4.1}$$

Here the components of the matrix $A(t, s)$ for a fixed t are functions with bounded variation in $[-\tau, 0]$ which are continuous with respect to t for a fixed s; $B(t)$, $C(t)$ are continuous matrices; the integral is to be understood as a Stieltjes integral. We assume that the sets P and Q of (1.2) are convex and that they depend continuously on t, i.e. that $P = P(t)$, $Q = Q(t)$; the set M is described by the condition $\{x\}_m \in M^0$, where M^0 is a convex, bounded, and closed set (the symbol $\{K\}_m$ denotes a matrix consisting of the first m rows of the matrix K). We assume that the strategies U and V are permissible, i.e. that the defining sets $U(t, x(s))$, $V(t, x(s))$ are convex, closed, and semicontinuous above relative to inclusion with respect to t and $x(s)$. The motions of systems (1.1) are now formalized within the framework of differential equations in contingencies with aftereffect. By a motion of system (1.1) from the position $p_0 = \{t_0, x_0(s)\}$, generated by a pair of strategies U, V we now mean any function $x[t] = x[t, p_0, U, V]$ continuous for $t \geqslant t_0$ which satisfies the condition $x[t_0 + s] = x_0(s)$ and (for almost all $t \geqslant t_0$) the inclusion

$$\frac{dx[x]}{dt} \in \int_{-\tau}^{0} dA(t,s)\, x_t[s] + B(t)\, U(t, x_t[s]) - C(t)\, V(t, x_t[s]) \tag{4.2}$$

(the existence of such solutions of system (4.2) by taking the limit of Euler broken curves [7]).

We denote by $\varepsilon_0(t, \sigma, x(s))$ $(\sigma \geqslant t,\ x(s) \in C_{[-\tau, 0]}$ the quantity

$$\varepsilon_0(t, \sigma, x(s)) = \max_{\|l\|=1} \Big[\int_t^\sigma \{\mu_2(\xi, \sigma, l) - \mu_1(\xi, \sigma, l)\} d\xi +$$

$$+ \rho_0(l) - l \Big\{ S(\sigma, t) x(0) + \int_{-\tau}^{0} dS_1(t, \sigma, s) x(s) \Big\}_m \tag{4.3}$$

provided the right side of (4.3) is nonnegative. Here l is an m-dimensional vector ; $\mu_1(\xi, \sigma, x(s)) = \max_u l\{S(\sigma, \xi)B(\xi)_m u,\ u \in P(\xi)$

$$\mu_2(\xi, \sigma, l) = \max_v l\{S(\sigma, \xi)C(\xi)\}_m v,\ v \in Q(\xi);\ \rho_0(l) = \min_y ly$$

$$y \in M^\circ;\ dS(\sigma, t)/dt = -\int_{-\tau}^{0} S(\sigma, t-s) dA(t, s),\ t < \sigma;\ S(t, t) = E$$

$$S(\sigma, t) = 0,\ t > \sigma;\ S_1(t, \sigma, s) = \int_0^\tau S(\sigma, t+\xi) A(t+\xi, s-\xi) d\xi$$

(we complete the definition of the function $A(t, s)$ for values $s < -\tau$ for each t in accordance with the equation $A(t, s) = A(t, -\tau)$ (see [14, 15]).

Let us denote by $\sigma = \vartheta^\circ$ the smallest root of the equation

$$\varepsilon_0(t_0, \sigma, x_0(s)) = 0, \qquad \sigma \geqslant t_0 \tag{4.4}$$

if this equation has a solution. Otherwise, we take ϑ° to represent an arbitrarily large number, larger than t_0.

Now let us assume that there exists a convex set $D(\sigma, t)$ such that the following conditions are fulfilled for any $t \in [t_0, \vartheta_0]$, $\sigma \in [t, \vartheta^\circ]$:

(1) $\{S(\sigma, t) B(t)\}_m P(t) = \{S(\sigma, t) C(t)\}_m Q(t) + D(t, \sigma)$;

(2) for any $u \in P(t)$ there exists a $v \in Q(t)$ such that

$$\{S(\sigma, t) B(t)\}_m u - \{S(\sigma, t) C(t)\}_m v \in D(\sigma, t).$$

(These conditions correspond to the conditions which occur in the problem of the minimax and maximin of time-optimal response in the case of ordinary systems [2, 8]).

We define the sets $U^*(t, x(s))$, $V_j^*(t, x(s))$ as follows:

$$U^*(t, x(s)) = \{u^e \mid q(t, \vartheta^\circ, x(s)) u^e = \max_{u \in P(t)} q(t, \vartheta^\circ, x(s)) u\} \tag{4.5}$$

if $\varepsilon_0(t, \vartheta^\circ, x(s)) > 0$; $\qquad U^*(t, x(s)) = P(t)),$ $\qquad$ if $\varepsilon_0(t, \vartheta^\circ, x(s)) = 0$

$$V_j^*(t, x(s)) = \{v_j^e \mid q_j(t, \vartheta^\circ, x(s)) v_j^e = \max_{v \in Q(t)} q_j(t, \vartheta^\circ, x(s)) v\} \tag{4.6}$$

if $p = \{t, x(s))\} \in G_j$; $\qquad V_j^*(t; x(s)) = Q(t),$ $\quad$ if $p \notin G_j$

Here $q(t, \vartheta^\circ, x(s)) = l^\circ(t, \vartheta^\circ, x(s)) \cdot \{S(\vartheta^\circ, t) B(t)\}_m$ and $l^\circ(t, \sigma, x(s))$ is the vector which maximizes (4.3) (this vector is unique by virtue of Condition (1));

$$q_j(t, \vartheta^\circ, x(s)) = \int_t^{\vartheta^\circ - \alpha_j} [\varepsilon_0(t, \sigma, x(s))]^{-2} l^\circ(t, \sigma, x(s)) \{S(\sigma, t) C(t)\}_m d\sigma$$

$$\alpha_j > 0, \qquad \{\alpha_j\} \to 0$$

the set $G_j = \{\{t, \ x(s)\} \mid \min_\sigma \varepsilon_0(t, \sigma, x(s)) > 0, \ \ t_0 \leqslant \sigma \leqslant \vartheta^\circ - \alpha_j\}$.

Theorem 4.1. Let Conditions (1) and (2) be fulfilled for system (4.1). If Eq. (4.4) has a solution, then the sequence $\{V_j^*\}$ of strategies defined by sets (4.6) is maximin; ϑ° is the value of the game. If Eq. (4.4) has no solutions, then the second-player strategy V_j^* ensures deviation of the motions of system (1.1) from the target M until an arbitrarily large instant ϑ_j.

This theorem can be proved in the same way as the analogous statements of [8]. The principal phase of the proof of optimality of the strategies V_j^* is the estimation of the right upper derived number $\limsup\limits_{\Delta t \to +0} (\Delta L_j \,/\, \Delta t)$ of the functional

$$L_j(t, x(s)) = \int_t^{\vartheta^\circ - \alpha_j} [\varepsilon_0(t, \sigma, x(s))]^{-1} d\sigma$$

along the motions of system (1.1); this number is given by

$$\limsup_{\Delta t \to +0} \left(\frac{\Delta L_j}{\Delta t}\right)_{(1.1)} = [\varepsilon_0(t, t, x_t[s])]^{-1} + \int_t^{\vartheta^\circ - \alpha_j} [\varepsilon_0(t, \sigma, x_t[s])]^{-2} l^\circ(t, \sigma, x_t[s]) \times$$
$$\times [\{S(\sigma, t) B(t)\}_m u[t] - \{S(\sigma, t) C(t)\}_m v_j^e[t] + \mu_2(t, \sigma, l^\circ(t, \sigma, x_t[s])) -$$
$$- \mu_1(t, \sigma, l^\circ(t, \sigma, x_t[s]))] d\sigma$$

Here $u[t]$ is the realization of the first-player control dictated by the strategy U; $v_j^e[t]$ is the realization of the second-player control dictated by the strategy V_j^*. Under Conditions (1) and (2) the control $v_j^e[t]$ in the domain G_j ensures the inequality

$$\limsup_{\Delta t \to +0} (\Delta L_j / \Delta t) \leqslant 0$$

whatever the first-player strategy U. We infer from this that the strategy V_j^* ensures deviation of the motions $x[t]$ of the system (4.1) from the target M at least until the instant $\vartheta_j = \vartheta^\circ - \alpha_j$.

Note 4.1. Theorem 4.1 implies that the set $W_t(\vartheta)$ of program absorption of the target M by system (4.1) at the instant ϑ (see [9]) consisting of those and only those $x(s) \in C_{[-\tau, 0]}$, which for any vector l satisfy the inequality

$$\int_t^\vartheta \{\mu_1(\xi, \vartheta, l) - \mu_2(\xi, \vartheta, l)\} d\xi - \rho_0(l) + \qquad (4.7)$$
$$+ l\{S(\vartheta, t) x(o) + \int_{-\tau}^0 dS_1(t, \vartheta, s) x(s)\}_m \geqslant 0$$

coincides under Conditions (1) and (2) with the positional absorption set $W(t, \vartheta)$ which plays a fundamental role in the basic text (Sects. 2 and 3) of the present study. We also note that (4.7) implies that the sets $W_t(\vartheta)$ are convex and closed in $C_{[-\tau, 0]}$.

Note 4.2. In the particular case where the set M° is a point $\{x\}_m = 0$, the set M is an $(n - m)$-dimensional linear subspace, whereupon Conditions (1) and (2) in the absence of aftereffect become the conditions occurring in [12], where, however, the problem of evasion is solved under the additional assumption of opponent discrimination.

In connection with Conditions (1) and (2) let us consider very briefly the conditions of regularizability of the problem of convergence with a set (see [2], Sect. 21) in the case of system (4.1) with aftereffect. The problem of convergence consists in constructing a first-player strategy U_0 which ensures that all the motions $x\,[t] = x\,[t,\ p_0,\ U_0,\ V_T]$ of system (4.1) hit the smallest possible ε°-neighborhood of the set M at a given instant ϑ.

If for any $t \in [t_0,\ \vartheta),\ x\,(s) \in C_{[-\tau,0]}$ which satisfy the condition $\varepsilon_0\,(t,\ \vartheta,\ x\,(s)) > 0$ (see (4.3)), the maximum in (4.3) is attained on a unique vector $l^\circ\,(t,\ \vartheta,\ x\,(s))$ (it is sufficient for Condition (1) to be fulfilled in order for this to happen) then, as in the case of ordinary systems [12], the problem can be solved on the basis of the extremal aiming rule [2]: the required strategy $U_0 = U^*$, where U^* is defined by sets (4.5) (we must set $\vartheta^\circ = \vartheta,\ \sigma = \vartheta$) in (4.5), where $\varepsilon^\circ = \varepsilon_0\,(t_0,\ \vartheta,\ x_0\,(s))$ (see also [11]). Such a case of the problem is called regular [2].

For nonregular cases of the convergence problem where the system is described by ordinary differential equations monograph [2] gives so-called regularizability conditions upon whose fulfillment the convergence problem can also be solved with the aid of the rule of generalized extremal aiming rule. It turns out that these regularizability conditions can be formulated in similar form in the case of system (4.1) with aftereffect.

These conditions are as follows: for any $t \in [t_0,\ \vartheta]$ there exists a convex set $R\,(t)$ such that (a) $\{S\,(\sigma,\ t)C\,(t)\}_m Q\,(t) = \{S\,(\sigma,\ t)B\,(t)\}_m P\,(t) + R\,(t)$; (b) for any $v \in Q\,(t)$ there exists a $u \in P\,(t)$ such that

$$\{S\,(\sigma,\ t)C\,(t)\}_m v - \{S\,(\sigma,\ t)\ B\,(t)\}_m\ u \in R\,(t).$$

Under conditions (a) and (b) there exists a strategy U which guarantees to the first player a result which is arbitrarily close to the best result equal to ε°. This statement can be proved by the method outlined in Sect. 21 of [2], on the basis of arguments similar to those used to prove Theorem 4.1.

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OPTIMIZATION OF THE TRACKING PROCESS WITH INFORMATION LAG

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Problems of optimal control with incomplete information are of considerable interest in connection with practical control problems. In the present paper we investigate the problem of optimizing the process of tracking an object in the case of incomplete and inexact data on its position. The errors in the measured data are due to: (1) information lag, (2) the presence of random disturbances in the measuring instruments. Certain assumptions enable us to educe the problem of determining the optimal tracking law to an ordinary optimal control problem. The optimal tracking law is obtained in explicit form for certain quality criteria.

1. Let the motion of the object under investigation be described by the system of differential equations

$$x^{\cdot}(t) = A(t)x(t) + f(t) \tag{1.1}$$

and let the tracking system have available to it the vector $y(t)$ given by

$$y(t) = \int_0^t Q(s)\,x(s-h)\,ds + \int_0^t \sigma(s)\,d\xi(s) \tag{1.2}$$

where the l-dimensional vectors $x(t)$ and $y(t)$ belong the the Euclidean space E_l.

Unless otherwise indicated, the vectors from E_l occurring below are to be understood as column vectors; The jth coordinate of a vector will be denoted by the same letter as the vector with the subscript j. For example, the vector $x(t) = (x_1(t), \ldots, x_l(t))'$; here and below primes indicate transposition.

We assume that the following restrictions on the coefficients of Eqs. (1.1), (1.2) are fulfilled throughout: the $(l \times l)$-dimensional matrices $A(t)$, $Q(t)$, $\sigma(t)$ and the vector $f(t) \in E_l$; the elements of $f(t)$ and $A(t)$ are continuous, and the elements of $\sigma(t)$ and $Q(t)$ are Borel-measurable and bounded; the constant $h \geqslant 0$; finally, $\xi(t)$ which is a Wiener random process, assumes values from E_l and has independent